LECTURE 20

ZIQUAN YANG

Today we introduce the last topic of the course, the divergence theorem.

Theorem 1. *Suppose that* $\mathbf{F}$ *is a continuously differentiable vector field defined on an open part of* $\mathbb{R}^3$ *which contains an oriented closed smooth surface* $\partial\Omega$ *enclosing a solid region* Ω*. Let* $\mathbf{n}$ *be the outward pointing unit normal vector field. Then we have*

$$\iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_{\Omega} \nabla\cdot\mathbf{F}\,dV. \tag{1}$$

Proof. Suppose that $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$. Then we have

$$\iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_{\partial\Omega} (M\mathbf{i}+N\mathbf{j}+P\mathbf{k})\cdot\mathbf{n}\,d\sigma = \iint_{\partial\Omega} M\mathbf{i}\cdot\mathbf{n}\,d\sigma + \iint_{\partial\Omega} N\mathbf{j}\cdot\mathbf{n}\,d\sigma + \iint_{\partial\Omega} P\mathbf{k}\cdot\mathbf{n}\,d\sigma. \tag{2}$$

A similar equation holds for the right hand side of (1). Therefore, to prove the theorem it suffices to prove it for the components $M\mathbf{i}, N\mathbf{j}$ and $P\mathbf{k}$ individually. Therefore, we might as well assume that $\mathbf{F} = P\mathbf{k}$.

Let us say Ω is of type-z if it is of the form

$$\{(x,y,z)\in\mathbb{R}^3 : (x,y)\in D, f(x,y)\le z\le g(x,y)\}$$

for some region D on the xy-plane and functions f and g. For any ε, we can find finitely many regions $\Omega_1,\cdots,\Omega_n$ in Ω of type-z such that both sides of (1) differ by at most ε if we replace Ω by the union of $\Omega_1,\cdots,\Omega_n$. Therefore, it suffices to assume that Ω is of type-z.

We only need to consider the cap and bottom of the region. The double integral on the cap becomes becomes

$$\iint_D P(x,y,g(x,y))\mathbf{k}\cdot(-g_x\mathbf{i} - g_y\mathbf{j} + \mathbf{k})\,dx\,dy = \iint_D P(x,y,g(x,y))\,dx\,dy.$$

Therefore, the left hand side of (1) becomes

$$\iint_D P(x,y,g(x,y)) - P(x,y,f(x,y))\,dx\,dy.$$

The right hand side of (1) becomes

$$\iiint_\Omega \frac{\partial P}{\partial z}\,dz\,dx\,dy = \iint_D \int_{f(x,y)}^{g(x,y)} \frac{\partial P}{\partial z}\,dz\,dx\,dy,$$

so we are done by the fundamental theorem of calculus. □

Example 2. Consider the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and the sphere S defined by $x^2+y^2+z^2 = a^2$ enclosing a solid ball Ω of radius a. Let us check divergence theorem in this case.

The outward unit normal vector $\mathbf{n}$ on the sphere is:

$$\mathbf{n} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}.$$

The dot product $\mathbf{F}\cdot\mathbf{n}$ is:

$$\mathbf{F}\cdot\mathbf{n} = (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\left(\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}\right) = \frac{x^2+y^2+z^2}{a} = \frac{a^2}{a} = a.$$

The surface integral becomes:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S a\,d\sigma = a\cdot\text{Area}(S) = 4\pi a^3.$$

Date: April 16, 2025.

The divergence of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3.$$

The volume integral is:

$$\iiint_{\Omega} \nabla \cdot \mathbf{F} \, dV = \iiint_{\Omega} 3 \, dV = 3 \times (\text{Volume of } \Omega) = 3 \times \frac{4}{3}\pi a^3 = 4\pi a^3.$$

Example 3. Find the flux of the vector field $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ outward through the surface of the cube bounded by the planes $x = 1$, $y = 1$, and $z = 1$ in the first octant.

Compute the divergence of $\mathbf{F}$:

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xz) = y + z + x$$

By the Divergence Theorem:

$$\text{Total flux} = \iiint_{\text{Cube}} (x + y + z) \, dV = \int_0^1 \int_0^1 \int_0^1 (x + y + z) \, dx dy dz = \frac{3}{2}$$